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A NOTE ON YIELD POINT PHENOMENON IN WAVE PROPAGATION

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Abstract—The propagation of stress pulses in a plastic material which exhibits delayed yielding and upper and lower yield points is considered by the method of expansion behind the wave front. The material model includes an evolutionary equation the presence of which is responsible for the yield phenomena and it is shown that it also leads to a very pronounced negative stress gradient behind the wave front.

INTRODUCTION

THE upper yield point and subsequent yield drop is a very familiar phenomenon in the tension testing of annealed mild steel. It is clearly related to the less familiar phenomenon of the yield delay time which arises when a stress in excess of the yield stress is instantaneously applied to a specimen and held constant. There is a delay in the development of plastic strain which may amount to several msec when the applied stress is around double the yield stress. The delay time appears to have been first observed by Hopkinson [1] and has been studied experimentally by a number of people, for example Clark and Wood [2], Krafft and Sullivan [3] and Campbell and Marsh [4]. It has never been clear how this yield delay time might affect the propagation of a stress wave in such a material. It is fairly easy to show that the rate independent (Karman–Taylor–Rakhmatulin) theory [5] of elastic plastic wave propagation cannot incorporate this effect. On the other hand the rate dependent (Malvern–Sokolovsky) theory [6], predicts that the decay in intensity of the wave front stress is given by the plastic strain rate immediately behind the wave. Thus any non-zero yield delay time would preclude an erosion of the stress wave. However a decay in the initial wave front with distance of propagation has been observed in waves of uniaxial strain in iron by Taylor and Rice [7]. Associated with this decay in the initial stress Taylor and Rice observed a stress drop behind the wave front before a smooth rise in stress to a maximum some distance behind the front. A stress drop behind the wave front has also been observed in commercially pure aluminum by Barker *et al.* [8], again under uniaxial strain conditions. This is surprising in that neither yield drop nor delay time effects appear to have been observed in aluminum. The time associated with the stress drop observed in [8] is around 6×10^{-8} sec and few apparata used in stress strain measurements have this degree of time resolution.

One of the most satisfactory explanations of the yield drop and the delayed yield phenomenon has been given in a series of papers by Gilman and his coworkers (see for example [9]). This explanation is based on the concept of dislocation multiplication. In the annealed material the dislocation density is low (10^6 cm/cm^3) and increases extremely rapidly with plastic strain. It is the purpose of this note to use the constitutive theory which arises from the dislocation multiplication approach in studying the propagation of stress waves in a material exhibiting the delayed yield phenomenon. For the purpose of this note the theory will be developed for rod waves merely to simplify the computations. The interested reader will have no difficulty in generating the appropriate solutions for uniaxial plane waves, spherical waves or cylindrical waves.

ANALYSIS

We consider here a straight thin rod modeled by the positive real line $0 \leq x < \infty$. The rod is initially undisturbed and a stress history $\sigma^*(t), t \geq 0$ is imposed on $x = 0$. The displacement from the initial position of the particle at point x will be denoted by $u(x, t)$ and the displacement gradient u_x by $\varepsilon(x, t)$. The mass density of the material will be assumed uniform in the initial configuration and denoted by ρ. The motion of the rod in regions where the displacement is defined and continuously differentiable is governed by the equation

$$\sigma_x = \rho u_{tt} \tag{1}$$

where $\sigma(x, t)$ is the stress and partial differentiation is indicated by subscripts. The response of the material will be based on the following model. The displacement gradient ε will be assumed divisible in two parts, ε^e, ε^p; the elastic part ε^e related to the stress through $\varepsilon^e = \sigma/Y$ with Y an elastic modulus and the plastic part ε^p given by a strain rate equation of the form

$$\varepsilon_t^p = bNv \tag{2}$$

where the three terms on the right hand side have the following physical interpretation; b is a Burgers vector, N a dislocation density and v a dislocation velocity. The density N is given by an evolutionary rate equation

$$\dot{N} = \frac{Nv}{\lambda}, \tag{3}$$

where λ is a mean free path for dislocation multiplication. Taken together these equations predict a linear increase in N with plastic strain when ε_t^p is one signed, i.e.

$$N = N_0 + 1/\lambda b \varepsilon^p. \tag{4}$$

The dislocation velocity v is generally taken to depend on stress and a number of different forms have been suggested; here we will use a power law form

$$v = v_0 \left(\frac{\sigma}{\sigma_0} \right)^n,$$

where v_0 and σ_0 are normalizing constants with the dimensions of velocity and stress respectively. When the exponent n is large (20–50) the material model is not very rate sensitive.

The response of the material is thus specified by the equations

$$\varepsilon^p = \varepsilon - \sigma/Y, \tag{5}$$

and

$$\varepsilon_t^p = bN_0\left(1+\frac{\varepsilon^p}{b\lambda N_0}\right)v_0\left(\frac{\sigma}{\sigma_0}\right)^n. \tag{6}$$

Equations similar to (5) and (6) have been studied in detail by Gilman [9] in application to certain quasi-static loading situations, and he has shown that this material model can predict a yield drop in constant strain rate tests and delayed yield in constant stress tests when appropriate values are given to the parameters N_0, λ, b and n.

A material model of this kind predicts an instantaneous elastic response and thus at a fixed position x the rod is at rest until a time $t-x/c$ where $c = (Y/\rho)^{\frac{1}{2}}$ is the (constant) wave velocity. We will be interested here in obtaining a solution for the stress at a fixed position as a Taylor series about the time of arrival,

$$\sigma(x,t) = \sum_{m=0}^{\infty}\frac{1}{m!}\left(t-\frac{x}{c}\right)^m \sigma_{t^m}\Big|_{t=x/c}$$

where the notation σ_{t^m} refers to $\partial^m\sigma/\partial t^m$. Solutions of this kind have been developed for the linearly viscoelastic material by Achenbach and Reddy [10] and it will be the purpose of this note to apply their technique to this model.

In what follows we will assume that $\sigma^*(t)$ has no discontinuities of any order for $t > 0$ and may be expanded as a MacLaurin series

$$\sigma^*(t) = \sum_{m=0}^{\infty}\frac{1}{m!}t^m\sigma_{t^m}\Big|_{t=0}.$$

It follows that at any time $t > 0$ the line $0 \le x < \infty$ is divided into two parts R^+; $0 \le x < ct$ and R^-; $ct < x < \infty$. Let $f(x,t)$ be a function defined and continuously differentiable in R^+, R^- which approaches definite limits f^+ and f^- as x approaches ct from within R^+ and R^- respectively. The jump in f across the common boundary point will be denoted and defined by

$$[f] \equiv f^+ - f^-.$$

The condition of compatibility of the propagating jump is, [11],

$$\frac{\mathrm{d}}{\mathrm{d}t}[f] = [f_t]+c[f_x]. \tag{7}$$

On the assumption that the displacement u is continuous we obtain

$$[u_t]+c[\varepsilon] = 0. \tag{8}$$

From this and the dynamical condition across a discontinuity of stress and particle velocity

$$[\sigma]+\rho c[u_t] = 0 \tag{9}$$

we obtain

$$\rho c^2 = \frac{[\sigma]}{[\varepsilon]} = Y, \tag{10}$$

where, to establish $[\varepsilon^p] = 0$ it has been noted that ε^p_t as a function of ε^p and σ is bounded for all finite values of these variables.

Following Achenbach and Reddy [10] we apply the condition of compatibility (7) to σ_{tm} and to $\rho c u_{tm+1}$, $m > 0$. Subtraction of the resulting expressions gives

$$\frac{d}{dt}[\sigma_{tm}] - \rho c \frac{d}{dt}[u_{tm+1}] = [\sigma_{tm+1} - Yu_{xtm+1}] + c[\sigma_{xtm} - \rho u_{tm+2}] \\ = -Y[\varepsilon^p_{tm+1}]. \tag{11}$$

Now

$$\frac{d}{dt}[\sigma_{tm-1}] = [\sigma_{tm}] + \rho c[u_{tm+1}],$$

so that

$$\rho c \frac{d}{dt}[u_{tm+1}] = \frac{d^2}{dt^2}[\sigma_{tm-1}] - \frac{d}{dt}[\sigma_{tm}]$$

leading to the final result

$$2\frac{d}{dt}[\sigma_{tm}] = \frac{d^2}{dt^2}[\sigma_{tm-1}] - Y[\varepsilon^p_{tm+1}]. \tag{12}$$

When $m = 0$ use of (9) with (11) gives

$$2\frac{d}{dt}[\sigma] = -Y[\varepsilon^p_t]. \tag{13}$$

It will be convenient to carry out the subsequent analysis in dimensionless variables. To this end we identify the characteristic time τ and the constant α associated with the rate equation (6)

$$\tau = \frac{\sigma_0}{YbN_0v_0}, \qquad \alpha = \frac{\sigma_0}{YbN_0\lambda}$$

and define $T = t/\tau$. Also we define

$$E = Y\varepsilon/\sigma_0; \qquad E^p = Y\varepsilon^p/\sigma_0; \qquad S = \sigma/\sigma_0; \qquad V = S^n.$$

The material response equations (5) and (6) become

$$E^p = (E - S)$$

$$E^p_T = (1 + \alpha E^p)V$$

and the decay equations (13) and (12) take the form

$$2\frac{d}{dt}[S] = -[E^p_T] \tag{14}$$

$$2\frac{d}{dt}[S_{Tm}] = \frac{d^2}{dT^2}[S_{Tm-1}] - [E^p_{Tm+1}] \tag{15}$$

with

$$[E^p] = 0. \tag{16}$$

SOLUTIONS

For $m = 0$ the solution is obtained as

$$S^+(T) = [S] = \frac{[S]^0}{(1+(n-1)([S]^0)^{n-1}T/2)^{1/n-1}} \tag{17}$$

where

$$[S]^0 = \sigma^*(0)/\sigma_0 = S^0.$$

For $m = 1$ equation (15) takes the form

$$2\frac{d}{dT}[S_T] = \frac{d^2}{dT^2}[S] - \alpha E_T^{p^+} S^{+n} - nS^{+n-1}S_T^+$$

or

$$2\frac{d}{dT}[S_T] + V_S(S^+(T))[S_T] = -\frac{1}{2}\frac{d}{dT}V(S^+) - \alpha V^2(S^+). \tag{18}$$

A suitable integrating factor for this equation is $e^{\xi/2}$ where

$$\xi = \int_0^T V_S(S^+(T'))\,dT' \tag{19}$$

in terms of which the solution takes the form

$$\begin{aligned} S_T^+ = [S_T] = e^{-\xi/2}[S_T]^0 - e^{-\xi/2}\int_0^T e^{+\xi(T')/2}\frac{1}{4}\frac{d}{dT'}V(S^+(T'))\,dT' \\ -\frac{\alpha}{2}e^{-\xi/2}\int_0^T e^{+\xi(T')/2}V^2(S^+(T'))\,dT'. \end{aligned} \tag{20}$$

We note that

$$\begin{aligned} \xi = \int_0^T V_S(S^+(T'))\,dT' &= -2\int_{S^0}^{S^+}\frac{V_S(S^+)\,dS^+}{V(S^+)} \\ &= -2\ln V/V^0 \quad \text{where } V^0 = (S^0)^n. \end{aligned}$$

Thus $e^{\xi/2} = V^0/V$ and after some manipulation we find that

$$[S_T] = \frac{V(S^+)}{V^0}[S_T]^0 - \frac{1}{4}V(S^+)\ln\frac{V(S^+)}{V^0} + \alpha V(S^+)(S^+ - S^0). \tag{21}$$

The computation has reached the point of diminishing returns at $n = 2$. The equation for arbitrary n is

$$2\frac{d}{dt}[S_{T^n}] + V_S^+[S_{T^n}] = \frac{d^2}{dT^2}[S_{T^{n-1}}] - G^n(T)$$

where

$$G^n(T) = \sum_{m=1}^{n-1}\binom{n}{m}\alpha(E^p_{T^{n-m}}|_{E^p=0})\sum_{k=1}^{m}(V_{S^k}|_{S=S^+})[S_{m-k}].$$

The formal solution is

$$[S_{T^n}] = \frac{V^+}{V^0}[S_{T^n}]^0 + \frac{1}{2}\frac{V^+}{V^0}\int_0^T \frac{V^0}{V^+}\frac{\mathrm{d}^2}{\mathrm{d}T'^2}[S_{T^{n-1}}]\,\mathrm{d}T'$$
$$-\frac{1}{2}\frac{V^+}{V^0}\int_0^T \frac{V^0}{V^+}G^n(T')\,\mathrm{d}T'$$

with $V^+ = VS^+(T)$ given by equation (17). For a step input, where $\sigma_t^*(0) = 0$, $\sigma^*(0) > 0$ formula (17) indicates that for $\alpha = 0$ the slope behind the wave front is positive for all T and if α and S_0 are such that $4\alpha S_0 \leq n$ the slope behind the wave front is again positive for all T. For $4\alpha S_0 > n$ the slope is initially negative but in all cases becomes positive for T large enough. It is interesting to note that although the decay curve for S is very insensitive to the value of S_0 (for values of n around 10 and over and $S_0 > 1$) the form of curve for S_T is very sensitive to the value of S_0.

In Fig. 1 the decay in S and the variation of S_T are shown for the case where $S_0 = 2$ with $n = 10$ and $\alpha = 100$. The implication of the solution with respect to the wave profiles at various times is also shown in this figure.

It is clear from Fig. 1 that the dislocation multiplication factor leads to an extremely rapid drop in the stress immediately behind the wave. It is known that the decay or amplification of a shock wave in a non-linearly elastic material is strongly influenced by the sign of the stress rate immediately behind the wave. The effect of elastic non-linearity will be of considerable importance in the case of plane wave propagation as in the plate slap experiment. Here the large volumetric deformation leads to non-linear elastic response. This effect has been discussed by Herrmann [13] for metals and by Schuler [14] for polymethyl methacrylate. This effect clearly delimits the range of applicability of the present results.

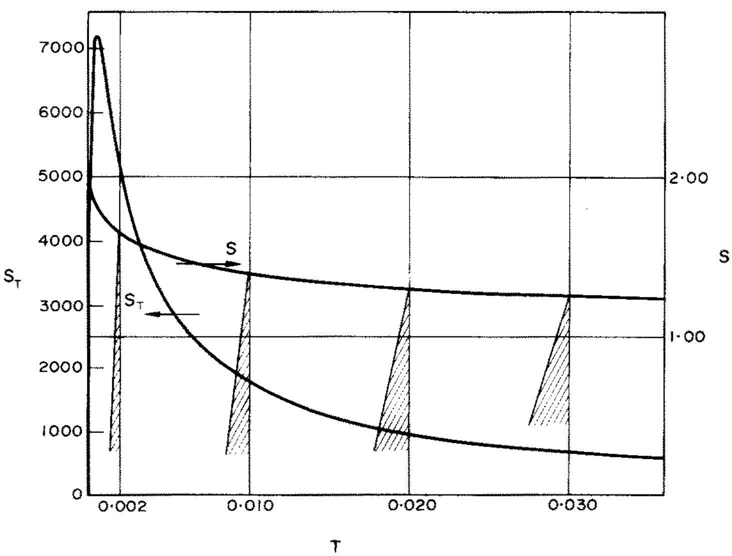

FIG. 1.

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Абстракт—Исследуется, методом разложения за фронтом волны, распространение импульса напряжения в пластическом материале, который проявляет запаздывание начала течения и верхней и нижней точек течения.

Модель материала заключает выеденное уравнение, наличие которого оказывается отвественным для явлений текучести. Оказывается, также, что это ведёт к самому определенному градиенту напряжений за фронтом волны.